International Journal of Theoretical Physics, Vol. 22, No. 4, 1983

Viscous Fluid in Static Spherically Symmetric Space-Time

S. R. Maiti

Department of Physics, Midnapore College, Midnapore-721101, India

Received October 8, 1981

The paper presents a static spherically symmetric viscous fluid solution of Einstein field equation, assuming an equation of state $p=(\gamma-1)\rho$. Though static, the solution has expansion, shear, and acceleration and can explain cosmological red shift. Also it has a particle horizon. The singularity at the origin and larger viscosity make it unfit to represent a real universe.

1. INTRODUCTION

Static space-time of Einstein was abandoned because of its inability to explain the observed red shift of light from distant galaxies and also that of de Sitter for its introduction of negative pressure. But the present paper proposes an investigation of the solution of the general relativity equation which admits a timelike hypersurface-orthogonal Killing vector and may be called static. However, the Killing vector is not parallel to the velocity vector of the cosmic matter so that there could be Doppler shift of light of one galaxy as observed from another. A somewhat similar study has been made recently by Florides (1980) on the Robertson–Walker metric and he has found that it admits a timelike Killing vector iff the metric is reducible to Minkowski or de Sitter form. Indeed this is true for any perfect fluid model in general relativity. If ξ^{μ} is a hypersurface-orthogonal Killing vector then (Cartar, 1972)

$$R_{\mu[\nu}\xi_{\lambda]}\xi^{\mu}=0 \tag{1}$$

For a perfect fluid source with pressure p, density ρ, and fluid velocity u^{α}, this reduces in terms of Einstein's equation to

$$(p+\rho)(u^{\mu}\xi^{\nu}-u^{\nu}\xi^{\mu})=0 \tag{2}$$

0020-7748/83/0400-0363$03.00/0

So one can have either (a) $p+\rho=0$, i.e., de Sitter or Minkowski form results, or (b) ξ^{α} and u^{α} are parallel, indicating vanishing of vorticity, shear, and expansion, and cannot explain red shift. Thus our study forces us to consider imperfect fluid with nonparallel ξ^{α} and u^{α}. The simplest form of imperfect fluid considered here is a viscous one and the space-time is assumed to be spherically symmetric.

2. STATIC VISCOUS FLUID WITH SPHERICAL SYMMETRY

The standard static spherical symmetric metric is

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{3}$$

with ν, λ as functions of r alone. Here $\xi^{\alpha} = \delta_0^{\alpha}$ and one may expect u^{α} to have at least one more component other than u^0. We use $0,1,2,3$ to represent t, r, θ, φ components. The energy-momentum tensor of the viscous fluid is

$$T_{\nu}^{\mu} = (p+\rho)u^{\mu}u_{\nu} - p\delta_{\nu}^{\mu} + 2\eta\sigma_{\nu}^{\mu} \tag{4}$$

with $u^{\alpha}u_{\alpha}=1$ and η is the coefficient of viscosity of the fluid and σ_{ν}^{μ} is the shear tensor defined as

$$\sigma_{\mu\nu} = u_{(\mu;\nu)} - \tfrac{1}{3}\theta h_{\mu\nu} - \dot{u}_{(\mu u \nu)} \tag{5}$$

and expansion θ and acceleration $\dot{u}_{\mu}$ are defined as

$$\theta = u^{\mu}_{;\mu} \tag{6}$$

$$\dot{u}_{\mu} = u_{\mu;\nu}u^{\nu} \tag{7}$$

one can then obtain from (4)

$$T^{\mu\nu}u_{\nu} = \rho u^{\mu} \tag{8}$$

The direct calculations from metric (3) demand $T_0^1 = T_0^2 = T_0^3 = 0$ and $T_2^2 = T_3^3$ in view of field equations of general relativity. So one can expect u^1 to be nonvanishing. Then from equations (4) and (8)

$$T_1^1 = T_0^0 = \rho \tag{9}$$

$$T_2^2 = T_3^3 = -\tfrac{1}{2}(\rho + 3p) \tag{10}$$

and $u^2 = u^3 = 0$. Thus the nonvanishing components of T_ν^μ are obtained by direct calculation from (3) as (cf. Landau and Lifshitz, 1962)

$$T_1^1 = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = \rho \tag{11}$$

$$T_2^2 = T_3^3 = -\frac{1}{2}e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) = -\frac{1}{2}(\rho + 3p) \tag{12}$$

$$T_0^0 = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} = \rho \tag{13}$$

From equations (11) and (13)

$$\nu' + \lambda' = 0 \tag{14}$$

and hence $\nu + \lambda = \text{const}$. Now from a scale transformation of t one can adjust

$$\nu = -\lambda \tag{15}$$

So the equations (11)–(13) reduce to

$$\rho = -e^{\nu}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{16}$$

$$\rho + 3p = e^{\nu}\left(\nu'' + \nu'^2 + \frac{2\nu'}{r}\right) \tag{17}$$

Consequently they give (cf. Maiti, 1982)

$$\rho' r = -3(p + \rho) \tag{18}$$

and

$$e^{\nu} = 1 - \frac{1}{r}\int \rho r^2 \, dr + \frac{B}{r} \tag{19}$$

Now if we have an equation of state $p = p(\rho)$ we can find ρ from (18) and get a solution from (19). Or if ρ is known explicitly, p and e^{ν} can be calculated.

We now consider an equation of state of the form

$$p = (\gamma - 1)\rho \tag{20}$$

with $1 \leqslant \gamma \leqslant 2$; then one can solve

$$\rho = A r^{-2\gamma} \tag{21}$$

$$e^{\nu} = 1 + \frac{A}{3(\gamma - 1)} r^{-(3\gamma - 2)} + \frac{B}{r} \qquad \text{for } \gamma \neq 1 \tag{22}$$

$$= 1 - \frac{\ln r}{r} + \frac{B}{r} \qquad \text{for } \gamma = 1 \tag{23}$$

with A and B constants.

3. SOME PHYSICAL QUANTITIES

The nonvanishing components of shear tensor are obtained from (5) and also from (4), (9), and (10) as

$$\sigma_1^1 = 2\left(\frac{1}{3}\theta - \frac{u^1}{r}\right) u^0 u_0 = \frac{1}{2\eta}(p + \rho) u^0 u_0 \tag{24}$$

$$\sigma_0^0 = 2\left(\frac{1}{3}\theta - \frac{u^1}{r}\right) u^1 u_1 = \frac{1}{2\eta}(p + \rho) u^1 u_1 \tag{25}$$

$$\sigma_2^2 = \sigma_3^3 = -\left(\frac{1}{3}\theta - \frac{u^1}{r}\right) = -\frac{1}{4\eta}(p + \rho) \tag{26}$$

$$\sigma_{10} = -2\left(\frac{1}{3}\theta - \frac{u^1}{r}\right) u_1 u_0 = -\frac{1}{2\eta}(p + \rho) u_1 u_0 \tag{27}$$

These equations are consistent only if

$$\theta = \frac{3u^1}{r} + \frac{3}{4\eta}(p + \rho) \tag{28}$$

Also the shear scalar is obtained from (24) to (27) as

$$2\sigma^2 = \sigma^{\alpha\beta}\sigma_{\alpha\beta} = \frac{3}{8\eta^2}(p + \rho)^2 \tag{29}$$

Integrating (28) one can have (considering η to be constant)

$$u^1 = \left(c - \frac{\rho}{4\eta}\right) r$$

Or,

$$u^1 = cr - \frac{A}{4\eta} r^{-(3\gamma-1)} \tag{30}$$

with c = const of integration. Again θ is calculated from (28) as

$$\theta = \frac{3(\gamma-1)A}{4\eta} r^{-3\gamma} + 3c \tag{31}$$

4. CHARACTERISTICS OF THE SPACE-TIME

The metric is singular at $r=0$, the pressure, density, expansion and shear scalar being infinite. Also u^1 approaches an infinite value as $r \to 0$ and the value is negative there. Such a singularity may be considered a sink. However, the metric is regular and analytic for $r > 0$.

As there are expansion, shear, and acceleration, the red shift of light of one galaxy as observed from another is considered as partly recessional and partly gravitational. However, the red shift is

$$1 + Z = \frac{dS_0}{dS_e} = \frac{(e^{\nu} - e^{-\nu} v^2)_0^{1/2}}{(e^{\nu} - e^{-\nu} v^2)_e^{1/2}} \tag{32}$$

where $v = dr/dt$. For observer $(v)_0 = 0$ and hence the red shift is

$$1 + Z = (e^{\nu/2})_0 (e^{\nu} - e^{-\nu} v^2)_e^{-1/2} \tag{33}$$

Also the solution has a particle horizon. If a light ray propagated radially from r_1 at t_1 reaching r_2 at t_2 then

$$\int_{t_1}^{t_2} dt = \int_{r_1}^{r_2} e^{-\nu}\, dr \tag{34}$$

Now the right-hand side of the integral is finite, hence as $t_1 \to 0$, the left-hand side converges and one can have a particle horizon.

The abnormal singularity at $r=0$ and larger viscosity make it unfit to represent a real universe. The other features like microwave background radiation and abundance of helium-deuteron have no ready explanation.

ACKNOWLEDGMENT

The author's grateful thanks are due to Professor A. K. Raychaudhuri of Presidency College, Calcutta for his helpful guidance.

REFERENCES

Cartar, B. (1972). "Black Hole Equilibrium States," in *Les Artres Occlus*. ed Gordon & Breach, New York.

Florides, P. S. (1980). *General Relativity and Gravitation*, **12**, 563.

Landau, L. D., and Lifshitz, E. M. (1962). *The Classical Theory of Fields*. Pergamon Press, New York.

Maiti, S. R. (1982), *Indian J. Phys*. (to be published).